

LINEAR CLIQUE-WIDTH FOR SUBCLASSES OF COGRAPHS, WITH CONNECTIONS TO PERMUTATIONS

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We prove that a hereditary property of cographs has bounded linear clique-width if and only if it does not contain all quasi-threshold graphs or their complements. The proof borrows ideas from the enumeration of permutation classes, and the similarities between these two strands of investigation lead us to a conjecture relating the graph properties of bounded linear clique-width to permutation classes with rational generating functions which would have far-reaching consequences if true.

1. INTRODUCTION

A set of graphs is referred to as a *property*, and a property which is closed downward under the induced subgraph ordering is called *hereditary*. While a bit nonstandard, for ease of notation *we use the term “class” to mean “hereditary property”* throughout this work.

A variety of measures of the “complexity” of graph classes have been introduced lately, and have proved very useful for algorithmic problems [4, 5, 7–9, 15, 16, 18, 20]. We are concerned with *clique-width*, introduced by Courcelle, Engelfriet, and Rozenberg [6], and more pertinently, the “linear” version of this parameter, introduced by Gurski and Wanke [12].

The *linear clique-width* of a graph G , $\text{lcw}(G)$, is the size of the smallest alphabet Σ such that G can be constructed by a sequence of the following three operations:

- add a new vertex labeled by a letter in Σ ,

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- add edges between all vertices labeled i and all vertices labeled j (for $i \neq j$), and
- relabel all vertices labeled i by j .

Such a sequence of operations is referred to as an *lcw expression* for G . An *optimal* lcw expression for G is one using exactly $\text{lcw}(G)$ labels. The distinction between linear clique-width and clique-width is that clique-width expressions allow a fourth operation: take the disjoint union of two (labeled) graphs previously constructed.

The *join* of two graphs G and H , denoted $G * H$, is formed from the disjoint union $G \cup H$ by adding all possible edges with one end in G and the other in H . We consider the following families of graphs in this work.

- The graph G is a *threshold graph* if, starting with K_1 , it can be built by repeatedly taking the disjoint union or join of a threshold graph with a single new vertex.
- The graph G is a *quasi-threshold graph* if, starting with K_1 , it can be built by repeatedly taking the disjoint union of two quasi-threshold graphs, or the join of a quasi-threshold graph with a single new vertex.
- The graph G is a *cograph* if, starting with K_1 , it can be built by repeatedly taking the disjoint union or join of two cographs.

Thus, rather trivially,

$$\text{threshold graphs} \subseteq \text{quasi-threshold graphs} \subseteq \text{cographs}.$$

The class of cographs is exactly the class of graphs with clique-width 2 (see [8]), but Gurski and Wanke [12] showed that it has unbounded linear clique-width.¹ This contrast is often cited in the literature to demonstrate that linear clique-width is a significant restriction of clique-width. However, that statement can be substantially strengthened: not only are the cographs not a minimal class of unbounded linear clique-width, but as our main result below shows, within the class of cographs it is possible to demarcate the precise border between bounded and unbounded linear clique-width.

Theorem 1.1. *Let \mathcal{C} be a class of cographs. Then \mathcal{C} has bounded linear clique-width if and only if it contains neither all quasi-threshold graphs nor the complements of all quasi-threshold graphs.*

We prove this result in the next two sections. For the remainder of the introduction we review several concepts used in our proofs.

Complements. In order to prove Theorem 1.1, we must show that both the class of quasi-threshold graphs and the class of their complements have unbounded linear clique-width. Section 2 is devoted to the proof of this claim for quasi-threshold graphs. The claim for their complements will then follow immediately from the following observation, mentioned in Gurski and Wanke [12].

Proposition 1.2. *For every graph G we have $\text{lcw}(\overline{G}) \leq \text{lcw}(G) + 1$.*

¹In fact, they show that cographs have unbounded *clique-tree-width*, a parameter that lies between linear clique-width and clique-width. It should be noted that the quasi-threshold graphs in fact have clique-tree-width 2.

Proof. Suppose that G has an optimal lcw expression over the alphabet Σ and that $\ell \notin \Sigma$. We can create a nearly optimal lcw expression for G over the alphabet $\Sigma \cup \{\ell\}$ in which every inserted vertex v is first inserted with the label ℓ . This is because we can then follow this insertion by immediately relabeling all vertices labeled ℓ , which will only be v , to whatever label the optimal lcw expression for G inserted v as. Moreover, at the moment we insert v in the optimal lcw expression, for every label $k \in \Sigma$, v must eventually be adjacent to either all vertices labeled k or no vertices labeled k . Thus in our new lcw expression over $\Sigma \cup \{\ell\}$ we may, immediately after inserting v with label ℓ , add edges between v and every currently existing vertex which v will eventually be adjacent to. Of course, we may instead add edges between v and every currently existing vertex which v will *never* be adjacent to, verifying that $\text{lcw}(\overline{G}) \leq \text{lcw}(G) + 1$. \square

Lettericity. Let Σ be an alphabet and $P \subseteq \Sigma \times \Sigma$ a set of pairs of letters. The *letter class* defined by (Σ, P) is the set of all graphs which are associated to words $w = w_1 \cdots w_n \in \Sigma^*$ by the following operation. First, create a vertex labeled w_1 . Then assuming that vertices corresponding to w_1, \dots, w_{k-1} have been constructed, create a new vertex labeled by w_k , and add an edge between this vertex and every vertex labeled w_i for which $(w_i, w_k) \in P$.

The *lettericity* of the graph G , $\text{let}(G)$, is then the least $|\Sigma|$ for which there is a set of pairs $P \subseteq \Sigma \times \Sigma$ such that G lies in the letter class defined by (Σ, P) . The lettericity of a class \mathcal{C} is the smallest $|\Sigma|$ for which there is a set of pairs $P \subseteq \Sigma \times \Sigma$ such that \mathcal{C} is contained in the letter class defined by (Σ, P) . Note that the lettericity of a class may be greater than the greatest lettericity of its members.

The concept of lettericity was introduced by Petkovšek [19], who observed that every class of bounded lettericity is well-quasi-ordered (this follows from Higman's Theorem [14]). For our purposes, it is important to note that the threshold graphs have lettericity 2 (let $\Sigma = \{u, j\}$ and $P = \{(u, j), (j, j)\}$; then the threshold graph G corresponds to the word $w = w_1 \cdots w_n$ where $w_k = u$ if the k^{th} vertex added to G was added by disjoint union and $w_k = j$ if it was added by join). We also make use of the following observation.

Proposition 1.3. *For all graphs G , $\text{lcw}(G) \leq \text{let}(G) + 1$.*

Proof. Suppose that G is associated to the word $w = w_1 \cdots w_n$ in the letter class defined by (Σ, P) . We construct an lcw expression for G using the labels $\Sigma \cup \{\ell\}$ where $\ell \notin \Sigma$. We begin by inserting a vertex labeled w_1 . Suppose then that we have inserted vertices corresponding to w_1, \dots, w_{k-1} , and that these vertices are now labeled by the letter of Σ that they correspond to and that all edges between them have been inserted. We now insert a new vertex labeled ℓ and add edges between all vertices labeled ℓ (only this new vertex) and all vertices labeled m for which $(m, \ell) \in P$. After this we relabel the vertex labeled ℓ by w_k . \square

Inflations. Let G be a graph on the labeled vertices $\{1, \dots, n\}$ and (H_1, \dots, H_n) a sequence of graphs. The *inflation* of G by H_1, \dots, H_n is the graph on the disjoint union $H_1 \cup \dots \cup H_n$ with additional edges added between $v \in H_i$ and $w \in H_j$ if and only if $i \sim j$ in G . We denote this graph by $G[H_1, \dots, H_n]$. Given classes \mathcal{C} and \mathcal{D} , the inflation of \mathcal{C} by \mathcal{D} , which we denote $\mathcal{C}[\mathcal{D}]$, is the set of all inflations $G[H_1, \dots, H_n]$ where G is a labeling of a graph from \mathcal{C} and $H_1, \dots, H_n \in \mathcal{D}$. The following result seems to be original.

Proposition 1.4. *For classes \mathcal{C} and \mathcal{D} of graphs, $\text{lcw}(\mathcal{C}[\mathcal{D}]) \leq \text{lcw}(\mathcal{C}) + \text{lcw}(\mathcal{D})$.*

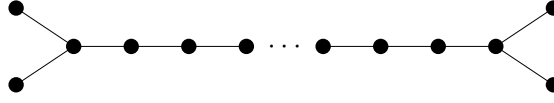


Figure 1: The antichain of split-end paths.

Proof. Consider any graph $G \in \mathcal{C}$ with n vertices labeled by $\{1, \dots, n\}$ and sequence (H_1, \dots, H_n) of graphs in \mathcal{D} . We need to find an lcw expression for $G[H_1, \dots, H_n]$ with at most $\text{lcw}(G) + \max\{\text{lcw}(H_i)\}$ labels. We build this expression by following an optimal lcw expression for G . Whenever this expression instructs us to insert the vertex $v_i \in G$, we instead use a disjoint set of $\text{lcw}(H_i)$ labels to create a copy of H_i . Once this copy of H_i is completed, we relabel all of its vertices to the label given to v_i in the lcw expression of G and continue. \square

Letting \mathcal{I} denote the class of all independent sets, note that $\text{lcw}(\mathcal{C}[\mathcal{I}]) = \text{lcw}(\mathcal{C})$; indeed, it is known [11, 13] that graphs of linear clique-width at most 2 are precisely the inflations of threshold graphs by \mathcal{I} . For the “regular” clique-width parameter we have $\text{cw}(\mathcal{C}[\mathcal{D}]) = \max\{\text{cw}(\mathcal{C}), \text{cw}(\mathcal{D})\}$, as shown by Courcelle and Olariu [8, Corollary 3.6].

Well-quasi-order. A class of finite graphs is said to be *well-quasi-ordered* if it contains neither an infinite antichain nor an infinite descending sequence of graphs (in the induced subgraph order). Because we are concerned only with finite graphs, our classes never contain infinite descending sequences of graphs, and thus well-quasi-order is synonymous with the lack of infinite antichains. Cographs are well-known to be well-quasi-ordered, so we will not have to worry about infinite antichains to prove Theorem 1.1. The following result allows us to use the technique of minimal counterexamples when proving results about well-quasi-ordered classes.

Proposition 1.5. *Let \mathcal{C} be a class of graphs. Then \mathcal{C} is well-quasi-ordered if and only if its subclasses satisfy the descending chain condition: every infinite descending chain of subclasses*

$$\mathcal{C} \supseteq \mathcal{D}_1 \supseteq \mathcal{D}_2 \supseteq \dots$$

contains a minimum element.

Proof. Suppose first that \mathcal{C} is well-quasi-ordered. If there were an infinite descending chain of subclasses of \mathcal{C} then we could find an infinite strictly descending chain,

$$\mathcal{C} \supsetneq \mathcal{D}_1 \supsetneq \mathcal{D}_2 \supsetneq \dots,$$

and then take $G_1 \in \mathcal{D}_1 \setminus \mathcal{D}_2$, $G_2 \in \mathcal{D}_2 \setminus \mathcal{D}_3$, and so on. If $i < j$ then G_i does not lie in \mathcal{D}_{i+1} , and thus also does not lie in \mathcal{D}_j , so G_j cannot contain G_i ; thus either $G_j \leq G_i$ or G_i and G_j are incomparable. However, this implies (by a simple application of Ramsey’s Theorem) that \mathcal{C} either contains an infinite descending sequence (impossible) or an infinite antichain of graphs, giving a contradiction.

For the other direction, if \mathcal{C} is not well-quasi-ordered, take $\{G_1, G_2, \dots\}$ to be an infinite antichain in \mathcal{C} . Then letting \mathcal{D}_i denote the set of graphs in \mathcal{C} that do not contain G_1, \dots, G_i gives an infinite descending chain, proving the result. \square

Finally, note that there is no general result connecting linear clique-width and well-quasi-ordering. The *split-end paths* shown in Figure 1 form an infinite antichain but have linear clique-width 3. Another example of this phenomena is the set of all cycles, which have linear clique-width 4.

Atomicity. We say that the class \mathcal{C} is *atomic* if it cannot be written as the union of two proper subclasses. This condition is equivalent to the *joint embedding property*: for every pair of graphs in \mathcal{C} there is another graph in \mathcal{C} which contains both. (Fraïssé [10] seems to have been the first to study these notions). We will need to make use of the combination of atomicity and well-quasi-order via the following proposition.

Proposition 1.6. *Every well-quasi-ordered class of graphs is a finite union of atomic classes.*

Proof. Suppose that \mathcal{C} is a well-quasi-ordered class. We build a complete binary tree whose root is labeled by \mathcal{C} . Consider a node labeled by \mathcal{D} . If $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ for proper subclasses \mathcal{D}_1 and \mathcal{D}_2 then we add children labeled by \mathcal{D}_1 and \mathcal{D}_2 to this node. Otherwise \mathcal{D} is atomic and nodes labeled \mathcal{D} in the binary tree are leaves. Because \mathcal{C} is well-quasi-ordered it does not have an infinite descending chain of subclasses (Proposition 1.5), so this tree contains no infinite paths and thus is finite by König's Lemma. Its leaves give the desired atomic classes. \square

2. QUASI-THRESHOLD GRAPHS HAVE UNBOUNDED LINEAR CLIQUE-WIDTH

Our proof that the quasi-threshold graphs have unbounded linear clique-width follows from a construction (detailed at the end of this section), which relies on a number of preliminary results.

First, without loss of generality, note that we can restrict our attention to lcw expressions in which labels can only be introduced by adding a new vertex with that label. We need two further basic observations about lcw expressions.

Proposition 2.1. *Every optimal lcw expression for a graph G has a step where all $\text{lcw}(G)$ labels are in use.*

Proof. Suppose that there is some optimal lcw expression where at most $\text{lcw}(G) - 1$ labels are ever in use simultaneously. To achieve a contradiction, we will modify the expression to remove a nominated label ℓ .

Following the lcw expression, we identify the point when label ℓ was first due to be included. There must be another label k not currently in use, and so we use label k in place of ℓ . If the original lcw expression later needs to insert a vertex with label k while k is still being used in place of ℓ , we can identify another label $k' \neq \ell$ not currently in use, and switch from using k to k' for ℓ immediately before the insertion of the vertex labeled k . By repeatedly switching to use a different label in place of ℓ whenever necessary, we build an lcw expression for G with only $\text{lcw}(G) - 1$ labels. \square

Proposition 2.2. *Fix an lcw expression for the graph G and a label ℓ which is used in this expression. Then there is an lcw expression for G in which the vertices are inserted in the same order, and where vertices labeled ℓ are never relabeled.*

Proof. Suppose that the given lcw expression does not preserve ℓ , and identify the first instance when vertices labeled ℓ are relabeled by some other label k . Instead, replace this operation by relabeling vertices labeled k to ℓ , and then swap all subsequent instances of ℓ and k . Finally, repeat this process until the desired expression is created. \square

In an lcw expression for a graph, a *sink label* is a label which is never used to create edges, and which is never relabeled by any other label. In other words, it is a label that can only be used for vertices whose neighbourhoods have been completed.

Proposition 2.3. *If every optimal lcw expression for a graph G is sink-free, then $\text{lcw}(G \cup G) = \text{lcw}(G) + 1$ and $G \cup G$ has an optimal lcw expression with a sink.*

Proof. First we prove that $\text{lcw}(G) + 1$ is a lower bound on $\text{lcw}(G \cup G)$. Suppose to the contrary that $G \cup G$ has an lcw expression with $\text{lcw}(G)$ labels. Denote the (vertices of the) two copies of G in $G \cup G$ by disjoint sets X and Y . By symmetry, we may assume that the first vertex which is inserted, say x , lies in X . Let ℓ denote the label of x when it is inserted. By Proposition 2.2, there is an lcw expression for $G \cup G$ using $\text{lcw}(G)$ labels in which x is still the first vertex inserted, and vertices labeled ℓ (specifically, the vertex x) are never relabeled.

Since the restriction of this lcw expression to Y induces an lcw expression for G , there is at least one point during the construction where every label is used to label some vertex in Y , by Proposition 2.1. Since x is labeled ℓ at this point of the construction, however, no edges may ever be added between a vertex of Y labeled ℓ and any other vertex of Y , as this would also create an edge between x and a vertex of Y . Therefore, in the restriction of this lcw expression to the vertices of Y , ℓ is a sink label, contradicting our hypotheses.

It is much easier to show that $\text{lcw}(G \cup G) \leq \text{lcw}(G) + 1$: use any lcw expression of G with $\text{lcw}(G)$ labels to create a copy of G , replace all of these labels by a new label, and then repeat the lcw expression of G to create a disjoint copy of G . Note that the lcw expression we have built for $G \cup G$ has a sink label, namely the new label we introduced. \square

Proposition 2.4. *Let G be an arbitrary graph and set $H = K_1 * (G \cup G)$. If G has at least one edge and an optimal lcw expression with a sink, then $\text{lcw}(G) = \text{lcw}(H)$ and every optimal lcw expression for H is sink-free.*

Proof. First we show that $\text{lcw}(G) = \text{lcw}(H)$. Obviously $\text{lcw}(H) \geq \text{lcw}(G)$, so we need only show that H has an lcw expression with $\text{lcw}(G)$ labels. We follow an optimal lcw expression with sink ℓ to build one copy of G , before relabeling all vertices of this copy with the label ℓ . We can then follow the same lcw expression to build a disjoint copy of G , and again relabel all vertices with ℓ . Since G has an edge, $\text{lcw}(G) \geq 2$, and thus G has at least one non- ℓ label. Insert a final vertex of this label and add edges between it and every vertex constructed before.

Now suppose for the sake of contradiction that H has an optimal lcw expression with a sink label ℓ . Denote the (vertices of the) two copies of G in H by disjoint sets X and Y and let z denote the remaining vertex of H . Using Proposition 2.1, suppose that when this lcw expression is restricted to X (resp., Y), the insertion of the vertex x (resp., y) marks the first point at which all labels are in use. By symmetry, we may assume that x is inserted before y . Thus when x is inserted, there is a vertex $x' \in X$ labeled by ℓ . Since ℓ is a sink label in the expression for H , x' can never acquire another edge. Therefore z must have been inserted before x' , and hence also before x . At the point x is inserted, there is a vertex $x'' \in X$ with the same label as z . Since x'' and z will always now have the same label, and x'' is inserted before y , it is now impossible to add the edge between y and z without adding an edge between y and x'' , a contradiction. \square

Although not necessary for our upcoming construction, note that for G not satisfying the hypotheses of Proposition 2.4, it is still possible to compute the linear clique-width of $H = K_1 * (G \cup G)$.

If G does not have an edge, then H is the inflation of P_3 by independent sets so $\text{lcw}(H) = 2$. In the final case, if all optimal lcw expressions for G are sink-free, then Proposition 2.3 shows that $\text{lcw}(H) \geq \text{lcw}(G \cup G) = \text{lcw}(G) + 1$. In fact, $\text{lcw}(H) = \text{lcw}(G) + 1$, as can be seen by following the lcw expression given in the proof of Proposition 2.3 to build $G \cup G$, then relabeling every vertex with the same label, inserting a new vertex with a different label, and adding all edges between this vertex and the rest of the graph.

Showing that the quasi-threshold graphs have unbounded linear clique-width is now easy. Set $G_1 = K_2$, and for $k \geq 1$ define G_{k+1} by

$$G_{k+1} = K_1 * (G_k \cup G_k \cup G_k \cup G_k).$$

Clearly these graphs are quasi-threshold graphs. By inductively applying Proposition 2.3 to G_k , and then inserting the outcome for $G_k \cup G_k$ into Proposition 2.4, we conclude that $\text{lcw}(G_k) = k + 1$ for all $k \geq 1$. Hence, quasi-threshold graphs do not have bounded linear clique-width.

Proposition 1.2 shows that $\text{lcw}(\overline{G}) \leq \text{lcw}(G) + 1$, so the class of complements of quasi-threshold graphs must also have unbounded linear clique-width as otherwise their complements (the quasi-threshold graphs) would have bounded linear clique-width, contradicting what we have just proved.

3. THE OTHER DIRECTION OF THEOREM 1.1

Having shown that the quasi-threshold graphs have unbounded linear clique-width, this section concerns the proof of the other direction of Theorem 1.1. Our proof is adapted from work of Albert, Atkinson, and Vatter [2] concerning permutations. The final section is devoted to speculation about further connections between linear clique-width and permutations.

Lemma 3.1. *Every atomic class \mathcal{C} of cographs is either*

- (1) *union-closed,*
- (2) *join-closed, or*
- (3) *contained in the inflation of the threshold graphs by a proper subclass $\mathcal{D} \subsetneq \mathcal{C}$.*

Proof. Let \mathcal{C} be a class of cographs and suppose that \mathcal{C} is neither union-closed nor join-closed. For every graph $G \in \mathcal{C}$, let \mathcal{U}_G^+ denote the set of graphs which are “unionable” with G in \mathcal{C} ,

$$\mathcal{U}_G^+ = \{H \in \mathcal{C} : G \cup H \in \mathcal{C}\},$$

and further define \mathcal{U}_G^- to be the set of graphs which are “at least as unionable” as G in \mathcal{C} ,

$$\begin{aligned} \mathcal{U}_G^- &= \{H \in \mathcal{C} : G \cup H' \in \mathcal{C} \implies H \cup H' \in \mathcal{C}\}, \\ &= \{H \in \mathcal{C} : H \cup H' \in \mathcal{C} \text{ for all } H' \in \mathcal{U}_G^+\}. \end{aligned}$$

First note that \mathcal{U}_G^+ and \mathcal{U}_G^- are both graph classes. Moreover, because \mathcal{C} is not union-closed, at least one of these must be a proper subset of \mathcal{C} for every graph G ; if G is unionable with every graph in \mathcal{C} then not every graph in \mathcal{C} can be as unionable as G . For every graph $G \in \mathcal{C}$, define

$$\mathcal{U}_G = \begin{cases} \mathcal{U}_G^+ & \text{if } \mathcal{U}_G^+ \neq \mathcal{C}, \text{ or} \\ \mathcal{U}_G^- & \text{otherwise (in which case } \mathcal{U}_G^- \neq \mathcal{C}). \end{cases}$$

In particular, note that every disconnected graph in \mathcal{C} has a connected component contained in some class \mathcal{U}_G .

Now note that passing from graphs to these classes reverses inclusions: thus if G is an induced subgraph of H ($G \leq H$) then H is at most as unionable as G ($\mathcal{U}_H^+ \subseteq \mathcal{U}_G^+$), and every graph which is as unionable as H is also as unionable as G ($\mathcal{U}_H^- \subseteq \mathcal{U}_G^-$). We claim that the number of *distinct* classes of the form \mathcal{U}_G^+ and \mathcal{U}_G^- (and thus also of the form \mathcal{U}_G) is finite. Suppose to the contrary that there are infinitely many distinct classes $\mathcal{U}_{G_1}^+, \mathcal{U}_{G_2}^+, \dots$ (the same argument will hold for classes of the form \mathcal{U}_G^-). Because the set of cographs is well-quasi-ordered, the set $\{G_1, G_2, \dots\}$ must contain an infinite ascending sequence

$$G_{i_1} \leq G_{i_2} \leq \dots$$

Our observation above then shows that

$$\mathcal{U}_{G_{i_1}}^+ \supseteq \mathcal{U}_{G_{i_2}}^+ \supseteq \dots$$

However, by the descending chain condition (Proposition 1.5), this sequence has a minimum element, contradicting our assumption that these classes are distinct. Thus we have established that there are only finitely many distinct classes \mathcal{U}_G , so there is a finite set $S \subseteq \mathcal{C}$ so that every class \mathcal{U}_G is of the form \mathcal{U}_H for some $H \in S$.

Define

$$\mathcal{U} = \bigcup_{G \in S} \mathcal{U}_G.$$

As each \mathcal{U}_G is a proper subclass of \mathcal{C} and \mathcal{C} is atomic, we know that \mathcal{U} is also a proper subclass of \mathcal{C} and that every disconnected graph in \mathcal{C} has a connected component in \mathcal{U} . By symmetry, there is also a proper subclass $\mathcal{J} \subsetneq \mathcal{C}$ such that every connected graph in \mathcal{C} contains a join-component in \mathcal{J} (because every cograph is either a disjoint union or a join).

It is now easy to establish by induction on $|G|$ that every graph $G \in \mathcal{C}$ is contained in the inflation of a threshold graph by graphs in $\mathcal{U} \cup \mathcal{J} \subsetneq \mathcal{C}$. This fact is clear for $|G| = 1$ because K_1 must be contained in $\mathcal{U} \cup \mathcal{J}$, and all larger graphs have either a component or join-component in $\mathcal{U} \cup \mathcal{J}$. \square

We are now in a position to complete the proof of our main theorem.

Proof of Theorem 1.1. Suppose to the contrary that the result is false. Because cographs are well-quasi-ordered, we may appeal to Proposition 1.5 to choose a minimal counterexample, say \mathcal{C} . Thus \mathcal{C} does not contain the class of all quasi-threshold graphs or their complements and has unbounded linear clique-width, but every proper subclass of \mathcal{C} has bounded linear clique-width. We now appeal to Lemma 3.1. If \mathcal{C} is contained in the inflation of the threshold graphs by a proper subclass of \mathcal{C} , then we are done by Proposition 1.4, so \mathcal{C} must be either union-closed or join-closed. By symmetry, suppose that \mathcal{C} is union-closed.

Let \mathcal{C}_c denote the set of graphs which are contained in a connected graph in \mathcal{C} ,

$$\mathcal{C}_c = \{G : G \leq H \text{ for a connected graph } H \in \mathcal{C}\}.$$

Clearly \mathcal{C}_c is a graph class, and if it were a proper subclass of \mathcal{C} then it, and thus also \mathcal{C} , would have bounded linear clique-width, a contradiction. Thus every graph in \mathcal{C} is contained in a connected graph in \mathcal{C} . We now show, by induction on $|G|$, that every quasi-threshold graph G lies in \mathcal{C} . The base case $|G| = 1$ is trivial because \mathcal{C} must be nonempty (\mathcal{C} has unbounded linear clique-width).

If $G = G_1 \cup G_2$, then by induction $G_1, G_2 \in \mathcal{C}$, and since \mathcal{C} is union-closed, $G \in \mathcal{C}$. Otherwise $G = G' * K_1$, and $G' \in \mathcal{C}$ by induction. Because $K_1 \in \mathcal{C}$ and \mathcal{C} is union-closed, $G' \cup K_1 \in \mathcal{C}$, and thus $G' \cup K_1$ is contained in a connected graph in \mathcal{C} . However, every (nontrivial) connected cograph is a join, and thus \mathcal{C} must contain $(G' \cup K_1) * K_1 \supseteq G$. This completes the proof of the claim, yielding the final contradiction to complete the proof of the theorem. \square

4. CONNECTIONS TO PERMUTATION CLASS ENUMERATION

In this section we explore the connection between linear clique-width and permutation patterns. First, we must give some terminology for the latter area. The permutation π of length n contains the permutation σ of length k (written $\sigma \leq \pi$) if π has a subsequence of length k which is order isomorphic to σ . For example, $\pi = 391867452$ (written in list, or one-line notation) contains $\sigma = 51342$, as can be seen by considering the subsequence $91672 (= \pi(2), \pi(3), \pi(5), \pi(6), \pi(9))$. Given a permutation π of length n , the corresponding *permutation graph* is the graph G_π on the vertices $\{1, \dots, n\}$ in which $i \sim j$ if $i < j$ and $\pi(i) > \pi(j)$. Note that if $\sigma \leq \pi$ then G_σ is an induced subgraph of G_π , but the reverse implication is not true in general. Given a permutation class \mathcal{C} , we define

$$G_{\mathcal{C}} = \{G_\pi : \pi \in \mathcal{C}\}.$$

The terminology used when studying this *subpermutation* order on permutations is, unfortunately, different from that used when studying the induced subgraph ordering. We have (mostly) chosen to adopt the conventions of both fields, so, for example, a hereditary property of permutations will be called a *permutation class*. The following lexicon gives further equivalents.

graph term	permutation term
induced subgraph	subpermutation
class / hereditary property	permutation class
minimal forbidden set of a class	basis
$\text{Free}(X)$	$\text{Av}(B)$
inflation	inflation
disjoint union	direct sum
join	skew sum
cograph	separable permutation
quasi-threshold graph	$\text{Av}(2413, 3142, 3412)$
complement of quasi-threshold graph	$\text{Av}(2413, 3142, 2143)$
threshold graph	permutation “drawn” on an X
bipartite permutation graph	321-avoiding permutation
class of bounded lettericity	geometrically griddable class

We alert the readers that the term “quasi-threshold graph” can be translated to the permutation context in several different ways. The lexicon lists the permutation equivalent as $\text{Av}(2413, 3142, 3412)$ because this is the class of all permutations whose permutation graphs are quasi-threshold. However, it is also true that $G_{\text{Av}(231)} = G_{\text{Av}(312)}$ is the entire class of all quasi-threshold graphs.

Despite the overlap in interests, the central questions in the two fields focus on different aspects, with permutational research generally possessing a much more enumerative flavour. Given a permutation class \mathcal{C} , we denote by \mathcal{C}_n the set of permutations in \mathcal{C} of length n . The *growth rate* of the

class \mathcal{C} is the limit superior of $\sqrt[n]{|\mathcal{C}_n|}$ as $n \rightarrow \infty$. We further refer to $\sum |\mathcal{C}_n| x^n$ as the *generating function* for \mathcal{C} . (Whether this generating function counts the empty permutation of length 0 is a matter of taste, and is immaterial to our discussion.) It is often of interest to characterize the generating functions of permutations classes as rational, algebraic, holonomic, etc. The decomposition we used in Section 3 to prove half of Theorem 1.1 was an adaptation of the decomposition used in the proof of the following enumerative result.

Theorem 4.1 (Albert, Atkinson, and Vatter [2]). *If \mathcal{C} is a subclass of the separable permutations that does not contain any of $\text{Av}(132)$, $\text{Av}(213)$, $\text{Av}(231)$ or $\text{Av}(312)$ then \mathcal{C} has a rational generating function.*

The other component of the proof of Theorem 4.1 was a result about *strongly rational permutation classes* — the class \mathcal{C} is said to be strongly rational if it and all its subclasses have rational generating functions. In this language, Theorem 4.1 states that a subclass of the separable permutations is strongly rational if and only if it does not contain $\text{Av}(231)$ or any symmetry of this class. It is well-known (and easy to prove) that $\text{Av}(231)$ has a nonrational generating function (these permutations are counted by the Catalan numbers), so this version of Theorem 4.1 is best possible.

It is tempting to wonder if the connection between bounded linear clique-width and strong rationality runs deeper. Let us begin with what is known, both positively and negatively. First, the permutational analogues of graph classes of bounded lettericity are known as *geometrically grid-dable classes*, and have received a significant amount of study recently. It is known that geometric grid classes are strongly rational [1], and that the inflation of a geometrically grid-dable class by a strongly rational class is also strongly rational [3]. Propositions 1.3 and 1.4 show that the corresponding graph classes have bounded linear clique-width.

Thus it is tempting to conjecture that strong rationality (of permutation classes) corresponds to bounded linear clique-width (of graph classes). However, this is a bit too strong. Elementary counting arguments show that strongly rational permutation classes must be well-quasi-ordered, but there are permutations which correspond to the infinite antichain of split-end paths, shown in Figure 1, and of bounded linear clique-width. Therefore bounded linear clique-width could only correspond to a weaker property than strong rationality. The permutation class \mathcal{C} is said to be *broadly rational* if every finitely based subclass of \mathcal{C} has a rational generating function (recall that “basis” is the term for the set of minimal forbidden objects in the study of the subpermutation order). We conjecture the following.

Conjecture 4.2 (The LCW Enumeration Conjecture). *The permutation class \mathcal{C} is broadly rational if and only if $G_{\mathcal{C}}$ has bounded linear clique-width. Moreover, \mathcal{C} is strongly rational if and only if $G_{\mathcal{C}}$ has bounded linear clique-width and is well-quasi-ordered.*

This conjecture would represent a very powerful tool for studying both linear clique-width and permutation class enumeration. For example, much of the work involved in proving Theorem 4.1 (concerning operations which preserve strong rationality) could have been avoided, as that result would follow from our Theorem 1.1. Going in the other direction, the 231-avoiding permutations have a nonrational generating function, so if Conjecture 4.2 were true then the work in Section 2 could have been avoided.

As another example, Albert, Ruškuc, and Vatter [3] give a lengthy, technical proof that the inflation of a geometric grid class by a strongly rational class is again strongly rational, a result which would follow quickly from Propositions 1.3 and 1.4, Conjecture 4.2, and a few standard results about well-quasi-ordering. Other consequences of Conjecture 4.2 would include the following.

- The inflation of one broadly rational permutation class by another is broadly rational.
- Every broadly rational permutation class which is well-quasi-ordered is actually strongly rational.
- Given a broadly rational permutation class \mathcal{C} , its class of *one-point extensions*, \mathcal{C}^{+1} , is also broadly rational (a one-point extension of a permutation is a permutation obtained by inserting a single entry, of any value and in any position; the analogous operation of adding a new vertex to a graph requires at most $\text{lcw}(G) + 1$ additional labels).
- If, as seems plausible, the quasi-threshold graphs and bipartite permutation graphs (see below) are the smallest (in some asymptotic sense) classes of permutation graphs of unbounded linear clique-width, then every permutation class of growth rate less than 4 is broadly rational.

Notably, all four of these implications have been conjectured by researchers studying permutation classes. Thus Conjecture 4.2 represents a possible unifying framework for permutation class enumeration.

We conclude with an implication of Conjecture 4.2 to the study of linear clique-width. In 2005, Albert and Atkinson made an unpublished conjecture that every proper subclass of $\text{Av}(321)$ is broadly rational. This class corresponds to the bipartite permutation graphs, which were shown by Lozin [17] to be a minimal class of unbounded clique-width. Could it be that the bipartite permutation graphs are actually a minimal class of unbounded *linear* clique-width?

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